Spectral representation of lattice gluon- and ghost propagators

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Outline

1. Introduction
   - Källén–Lehmann representation
   - Tikhonov Regularization
2. Toy Models
   - Breit-Wigner model
   - $\int_0^\infty \rho(\omega)\omega \, \mathrm{d}\omega = 0$ model
3. Lattice Data
   - Gluon Propagator
   - Ghost Propagator
4. Conclusion/Outlook

Introduction

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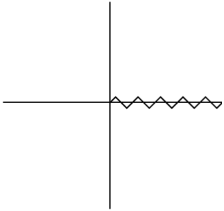

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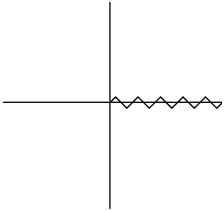

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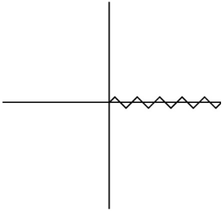

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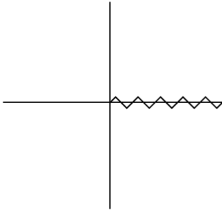

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- Therefore, lattice simulations give $G(p_4)$, for $p_4^2 > 0$.
- Analytically this has a unique analytical continuation to Minkowski $p_4^2 < 0$.
- Numerically this is very sensitive to noise: *ill-defined*.

Källén–Lehmann representation

Strategy: find the Källén–Lehmann representation.

$$G(p_4) = \int_{-\infty}^{\infty} \frac{\rho(\omega)}{\omega - ip_4}\,\mathrm{d}\omega$$

Where $\rho(-\omega) = -\rho(\omega)$.

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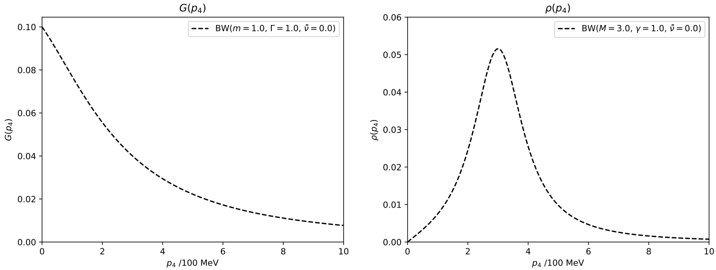
Where $\rho(-\omega) = -\rho(\omega)$. The challenge: find $\rho(\omega)$ for given lattice data p_4 vs $G(p_4)$.



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- Therefore, we expect positivity violations! Specifically:

$$\int_0^\infty \omega\rho(\omega)\,\mathrm{d}\omega = 0$$

Inverting G: Naive approach

$$G(p_4) = \int_{-\infty}^{\infty} \frac{\rho(\omega)}{\omega - ip_4}\,\mathrm{d}\omega = \int_{-\infty}^{\infty} K(p_4,\omega)\rho(\omega)\,\mathrm{d}\omega\,, \quad K(p_4,\omega) := \frac{1}{\omega - ip_4}$$

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Problem

$\mathbf{K}$ has small singular values, making it ill-conditioned on the numerical level $\to$ ill-defined inversion problem

Sophisticated Approaches

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when including the covariance matrix Σ.

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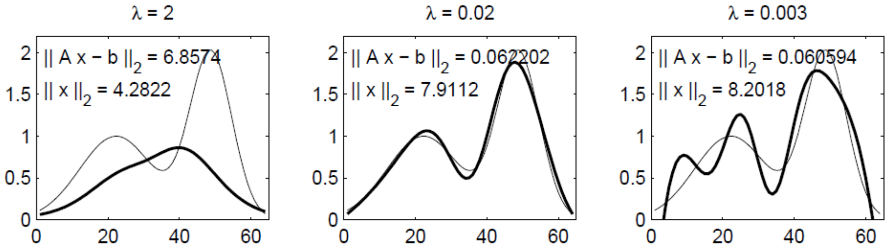
– No unique criterion to choose λ.

How to select the optimal λ?



Too small lambda values mean no regularization at all: over-fitting. Too large results in over-smoothing.

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We therefore minimize

$$J_\lambda = \left\|\mathbf{K}\vec{\rho} - \vec{G}\right\|^2 + \lambda\|\vec{\rho}\|^2 \tag{2}$$

subject to $\left\|\mathbf{K}\vec{\rho} - \vec{G}\right\|^2 = \|\vec{\sigma}\|^2$. This has a unique solution.

Tikhonov Regularization

$$J_\lambda = \sum_{n=1}^{N} \frac{1}{\sigma_n^2} \left| \int_{-\infty}^{\infty} \frac{\rho(\omega)}{\omega - ip_n}\, \mathrm{d}\omega - G_n \right|^2 + \lambda \int_{-\infty}^{\infty} \rho(\omega)^2\, \mathrm{d}\omega$$

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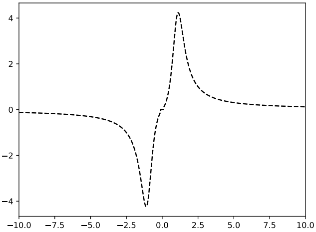
For a given λ

$$\rho(\nu) = -\frac{1}{\lambda} \sum_{n=1}^{N} \frac{1}{\sigma_n^2} \frac{c_n}{\nu - ip_n} \tag{3}$$

Toy Models

Reconstruction Process

$\rho^{\text{orig}} \to G_n^{\text{orig}} \to G_n \in \mathcal{N}(G_n^{\text{orig}}, (\epsilon G_n^{\text{orig}})^2)$



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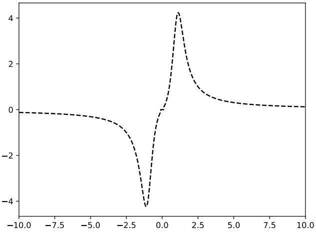

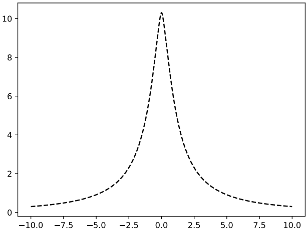

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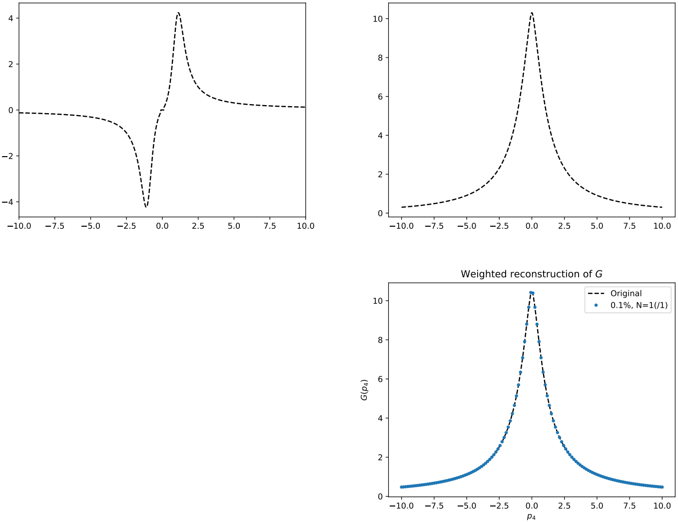

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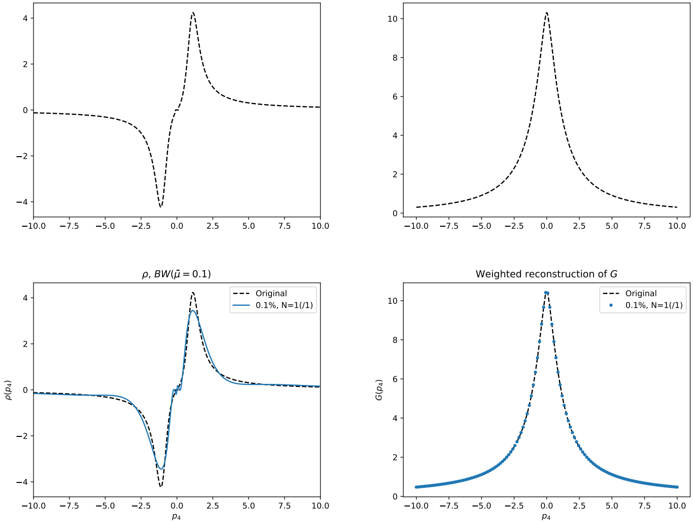

BW-model reconstruction

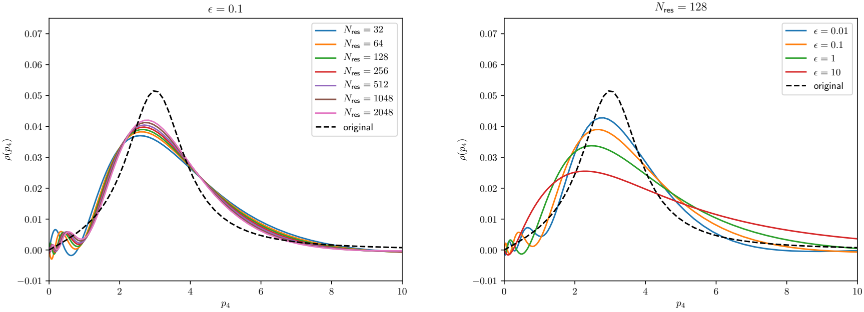

Figure: $BW(M = 300\text{MeV}, \gamma = 100\text{MeV}, \bar{\nu} = 0)$

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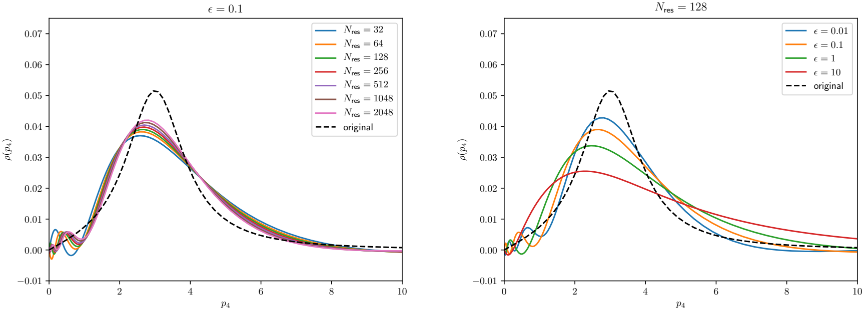

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Integrals are preserved within statistical error with Tikhonov reconstruction

Quality of the fit

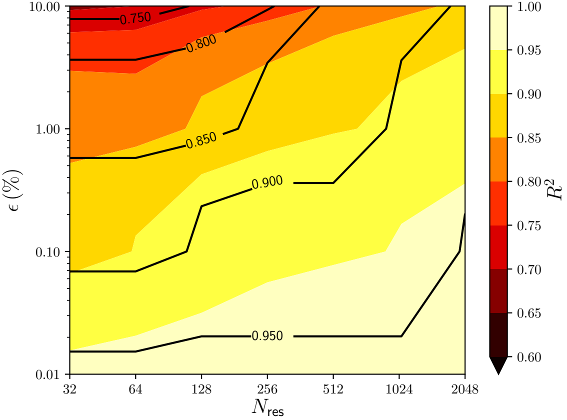

Figure: R^2 for the quality of the fit

$\int_0^\infty \rho(\omega)\omega \,\mathrm{d}\omega = 0$ model

For gluons we expect the following sum rule:

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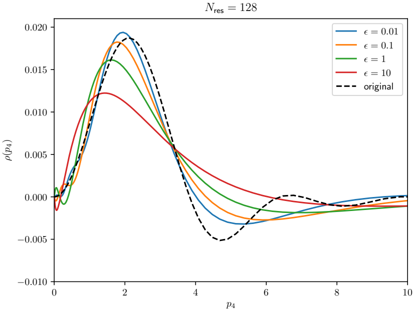
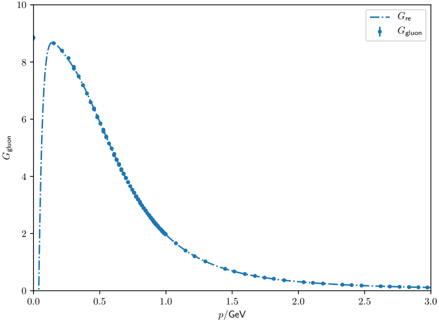


Lattice Data

Observation

$T = 0$ Gluon propagator data, 80^4 lattice, $\beta = 6.0$, normalised at 3GeV



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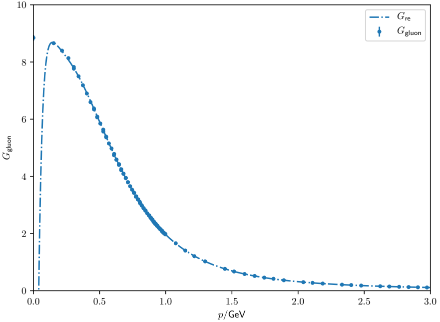

Figure: Gluon Propagator

- The datapoint at $p_4 = 0$ is very poorly reconstructed.
- Pragmatic solution:

$$\int_{-\infty}^{\infty} \rightarrow \left(\int_{-\infty}^{-\omega_0} + \int_{\omega_0}^{\infty} \right)$$

ω_0 vs α

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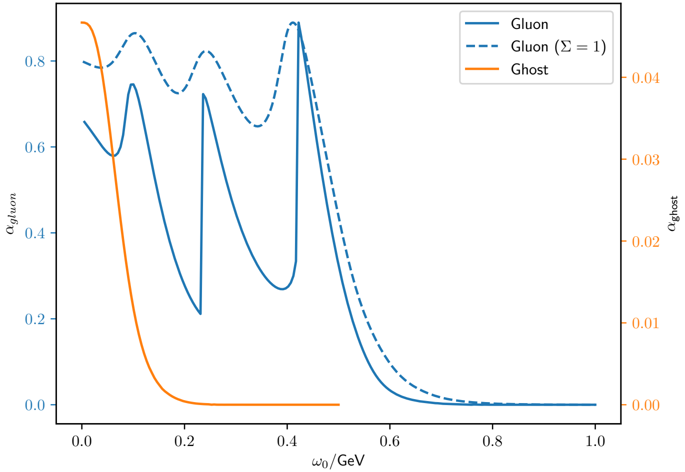

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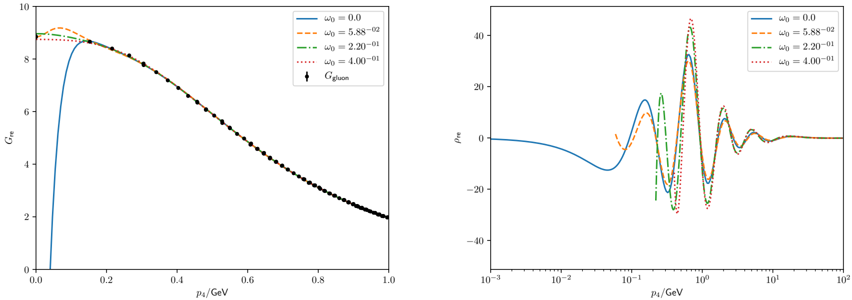

Figure: Left: Gluon propagator. Right: Gluon spectral density

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- Numerical inversions don't like δ-peaks.
- It can be avoided by inverting $g(p) = p^2 G(p)$ instead.

Ghost Propagator

$T = 0$ Ghost propagator data, 80^4 lattice, $\beta = 6.0$, normalised at 3GeV

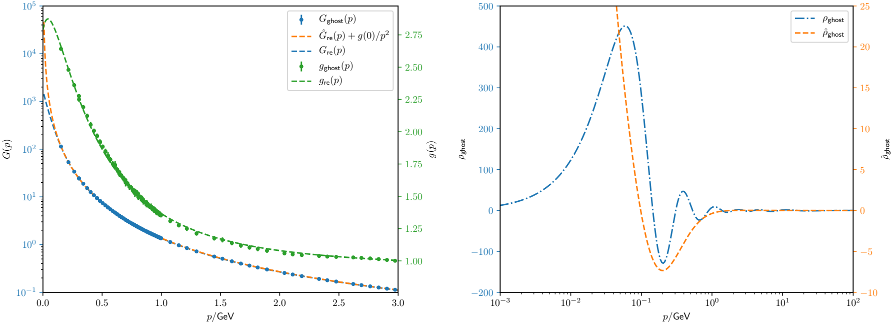

Figure: Left: ghost propagator. Right: ghost spectral representation

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+ Study $T > 0$ datasets, especially around the critical temperature

Dudal, David, Orlando Oliveira, and Paulo J. Silva (2014). "Källén-Lehmann spectroscopy for (un)physical degrees of freedom". In: *Phys. Rev.* D89.1, p. 014010. DOI: 10.1103/PhysRevD.89.014010. arXiv: 1310.4069 [hep-lat].

Tripolt, Ralf-Arno et al. (2018). "Numerical analytic continuation of Euclidean data". In: arXiv: 1801.10348 [hep-ph].

Roelfs et al. 2018?

ip-Formalism, weighted vs unweighted

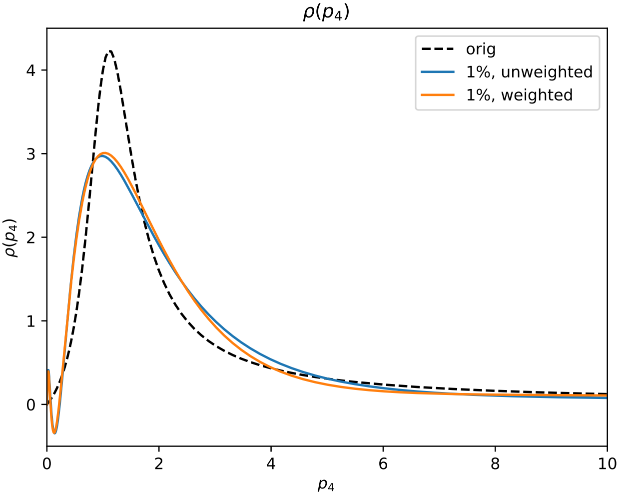

Figure: Weighted vs unweighted, resp. $N_p = 256$

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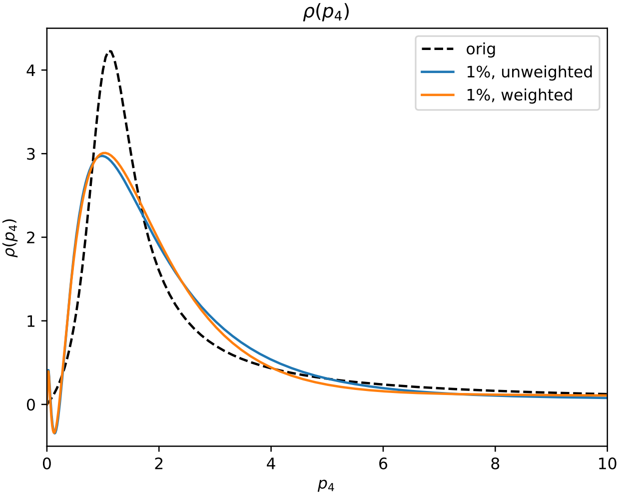

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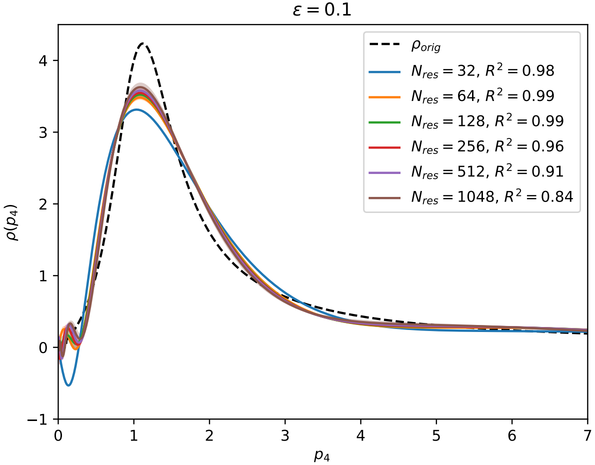
Weighted performs better in the UV, use the weighted algorithm from now

Reproducibility/Stability

Does every sample $G_n \in \mathcal{N}(G_n, (\epsilon G_n)^2)$ give approximately the same λ?



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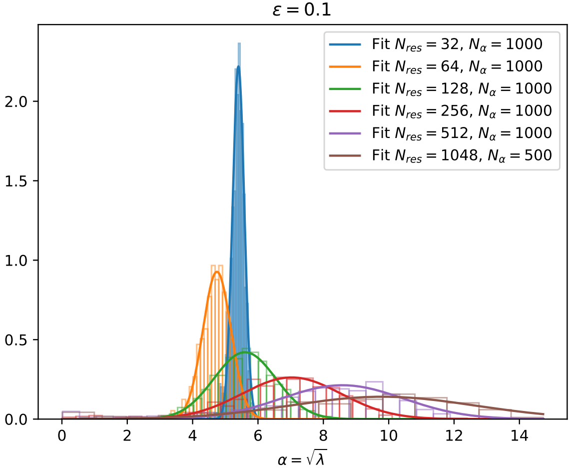

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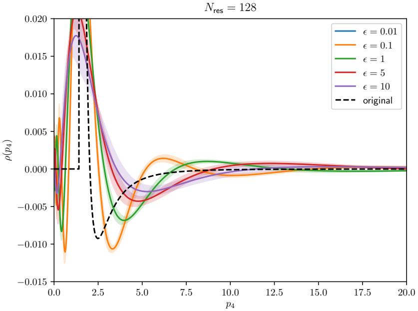
Toy model

$$\rho(\omega) = \begin{cases} \frac{-1}{\omega^4+4} + \frac{A}{\omega^6+2} & \omega > \sqrt{2} \\ \rho(-\omega) = -\rho(\omega) \end{cases}$$



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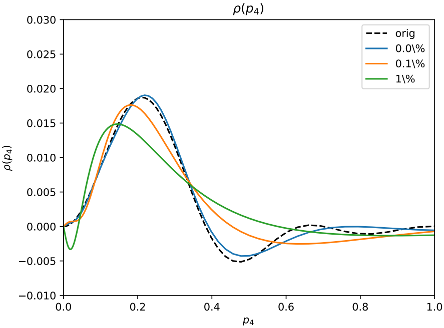

$\int \rho(\omega)\omega \, \mathrm{d}\omega = 0$ model, ip vs p^2