"Testing General Relativity: matter in... action!"

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Testing

Finding observables, searching for smoking guns.

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General Relativity

Theories of gravity.

Testing

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General Relativity

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Matter

Compact objects, configurations.

Testing

Finding observables, searching for smoking guns.

General Relativity

Theories of gravity.

Matter

Compact objects, configurations.

Action

Math and orbits.

Scattering processes

Questions and motivations

Is the environment affecting gravitational waves propagation?

Which is the effect of an impinging wave hitting on a binary system?

Is this phenomenon sharing some aspects with the classic electromagnetic counterpart?

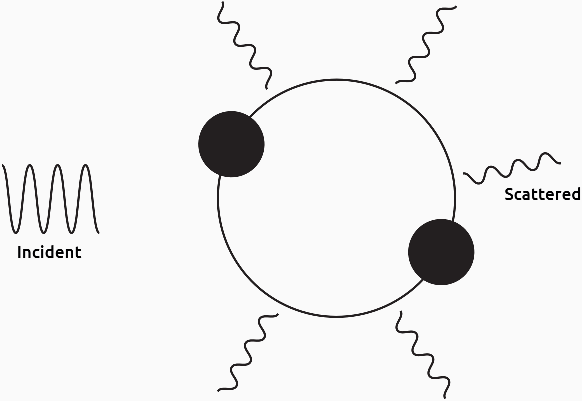


Figure 1: Scattering setup. [LA, Bernard, Blas, Cardoso: to be published (2018)]

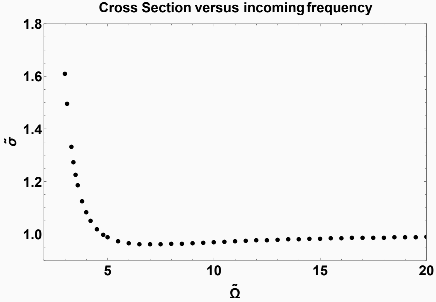
EM Review

Perturbed dipole: linear perturbation theory.

Circular orbit: **resonant frequency** appears for $\Omega = 2\omega_0$.

Scattered radiation: $\vec{E}^{\text{scattered}} = \vec{E}_\omega + \vec{E}_{2p}\ (+\vec{E}_{\text{LW}})$.

Scattered cross section: in the high frequency limit, it agrees with standard, classical results.



Gravitational wave scattering

Preliminaries

Incoming monochromatic GW at a frequency Ω:

$$H_{ij} = H_+ \cos\left(\Omega\, t - k\, z\right)\, e^+_{ij} + H_\times \sin\left(\Omega\, t - k\, z\right)\, e^\times_{ij}\,.$$

Einstein equation:

$$\Box l^{\mu\nu} = \frac{16\pi G}{c^4} T^{\mu\nu} + \Lambda^{\mu\nu}\,, \quad l^{\mu\nu} = h^{\mu\nu} + H^{\mu\nu}.$$

Post-Newtonian iteration of the field equations to find the potential entering the space-time metric of the system.

$$h^{00ii} = -\frac{4V}{c^2}, \quad \Box V = -4\pi G\sigma - H^{ab}\partial_{ab}h^{00ii}$$

$$V_N = \frac{Gm_1}{r_1} + \frac{Gm_2}{r_2}, \;\; V_h = \Delta^{-1}\left[-H^{ab}\partial_{ab}\left(\frac{Gm_1}{r_1} + \frac{Gm_2}{r_2}\right)\right].$$

Framework

The **geodesic equations** give us the acceleration felt by each particles.

From the acceleration to the **Lagrangian** and then to the **Hamiltonian**.

Conservation laws: linear momentum, angular momentum, variation of energy etc.

Methodology

Approach based on **angle-action** variables

$$(r, \theta, \varphi,\ p_r, p_\theta, p_\varphi) \Longrightarrow (J_i, \theta_i)$$

Perturbation theory in angle-action variables to describe the evolution of the perturbed Keplerian problem.

It is possible to relate the new set of variables to the **orbital elements** (a, ι, etc.)

Results...

Evolution of the **Keplerian parameter**: wave parallel and perpendicular to the orbital plane.

Resonances appearing when the frequency of the incoming wave is an integer multiple of the orbital frequency.

The scattering cross section can be computed, generically for any binary orientation and any incoming gravitational wave.

...and more results

In the high frequency limit, the cross section for either edge- or head-on configurations is the same,

$$\sigma_+ = \frac{4624\,\pi^2 c^2 \nu\, x^{5/2}}{315\,\omega_0^2}\,\frac{\omega_0}{\Omega}\,,$$

where $x \equiv \left(\frac{Gm\omega_0}{c^3}\right)^{2/3}$.

Scattered gravitational wave: the ratio between the scattered-wave amplitude and the incident wave can be of order 10^{-5} for known pulsars.

Modifications to GR

Lovelock's theorem

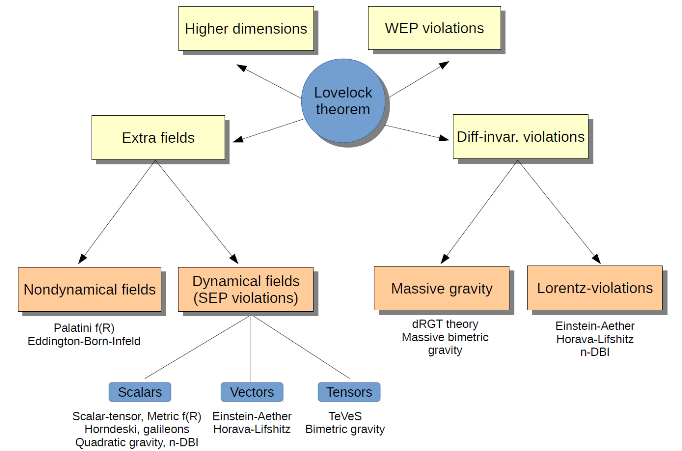


Figure 2: Zoo of modified theories of gravity. [arXiv/gr-qc:1501.07274]

Vector-Tensor theories

Framework

Action for vector-tensor theory

$$S_{[R,X^\mu]} = \frac{1}{16\pi}\int d^4x\sqrt{-g}[R - F_{\mu\nu}F^{\mu\nu} + \omega X_\mu X^\mu R + \eta X^\mu X^\nu R_{\mu\nu}]$$

Equation of for the vector fiels,

$$F^{\mu\nu}_{;\nu} - \frac{1}{2}\omega R X^\mu - \frac{1}{2}\eta X^\nu R^\mu_\nu = 0$$

Questions

For high compact objects, is here a vectorized solution?

Which is the nature of the instability that drives these new solutions?

Stability of flat stars ($\eta = 0$)

Let's linearize the equation:

$$F^{\mu\nu}_{;\nu} + 4\pi G \omega X^{\mu} T = 0\,,$$

and decompose the perturbation:

$$\sum_{l,m}\left(\begin{bmatrix} 0 \\ 0 \\ a_{lm}(t,r)(\sin\theta)^{-1}\partial_\phi Y_{lm} \\ -a_{lm}(t,r)\sin\theta\partial_\theta Y_{lm}\end{bmatrix} + \begin{bmatrix} f_{lm}(t,r)Y_{lm} \\ h_{lm}(t,r)Y_{lm} \\ k_{lm}(t,r)\partial_\theta Y_{lm} \\ k_{lm}(t,r)\partial_\phi Y_{lm}\end{bmatrix}\right)\,,$$

Master equation for axial sector:

$$F(r)G(r)a''_{lm} + \frac{1}{2}a'_{lm}\left(G(r)F'(r) + F(r)G'(r)\right)$$
$$+ a_{lm}\left(\omega^2 - \frac{F(r)\left(l(l+1) + 4\pi r^2 \Omega(3p(r) - \epsilon(r))\right)}{r^2}\right) = 0\,.$$

Axial (in-)stability results

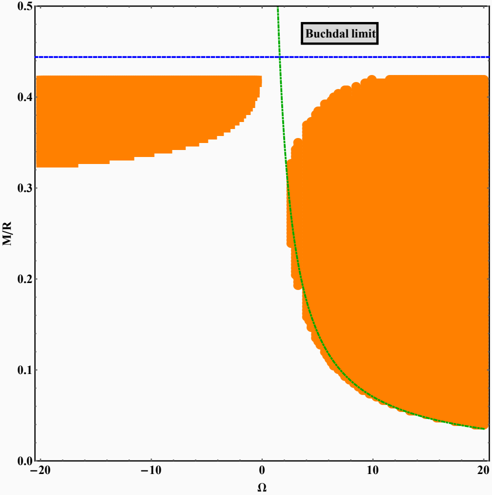


Figure 3: Unstable solutions for flat stars in the phase-space. [LA, Cardoso, Gualtieri (in preparation, 2018)]

Methodology for vectorized stars

Find the **modified Tolman–Oppenheimer–Volkoff equation** (TOV): system of equations concerning with the structure of a spherically symmetric body in gravitational equilibrium.

Integration of the TOV from the center to the boundary (radius) of the star and from there to spatial infinity.

Vectorized solution will be given by a non trivial value of the vector field inside the star.

Results

Final considerations

The study of the **scattering process** gives us a first idea on the **interaction** between astrophysical system and gravitational waves.

Via **precision GW astronomy** it will become more important to have models to understand these interactions.

The possibility to have **non trivial configurations** for neutron stars or black holes lead us to be endowed with a tool to distinguish between the different theories of gravity.

Scattering of scalar, electromagnetic and gravitational waves from binary systems

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What’s next?

Future...

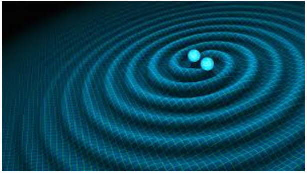
Vector-tensor solutions.

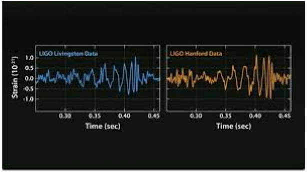
Generalization toward a general Einstein-Proca theory?

Inclusion of a scalar field? Black hole solution?

Consequences of GW scattering?

Stars as detectors: GW noise background?





Appendix

Classic dipole physics

Action:

$$S = \int d^3x dt \left[-\frac{F_{\mu\nu} F^{\mu\nu}}{4\mu_0} - A^1_\mu J^2_\mu - A^2_\mu J^1_\mu \right] + c^2 \int d\tau \left(m_1 + m_2 \right).$$

Radial equation:

$$\ddot{r} - \frac{4L^2}{M^2 r^3} = -\frac{2q^2}{Mr^2}$$

where $L = Mr^2 \dot{\phi}/2$.

Dipole definition:

$$\vec{d} = q_1 \vec{r}_1 + q_2 \vec{r}_2 = \frac{q_1 m_2 - q_2 m_1}{m_1 + m_2} \vec{r} = \mu \left(\frac{q_1}{m_1} - \frac{q_2}{m_2} \right) \vec{r},$$

Differential intensity emitted:

$$dI = c \frac{|\vec{H}|^2}{4\pi} R_0^2 do \to I = \frac{2}{3c^3} |\ddot{\vec{d}}|^2.$$

Electromagnetic scattering framework 1

Perturbation vector potential:

$$A_1^\mu \to A_1^\mu + A_\omega^\mu = \left(\frac{\Phi_1}{c} + \frac{\Phi_\omega}{c}, \vec{A}_1 + \vec{A}_\omega\right),$$

$$A_2^\mu \to A_2^\mu + A_\omega^\mu = \left(\frac{\Phi_2}{c} + \frac{\Phi_\omega}{c}, \vec{A}_2 + \vec{A}_\omega\right).$$

Equation of motion:

$$\begin{aligned}\ddot{\vec{R}}_{\mathrm{CM}} &= \frac{q_1+q_2}{m}(\vec{E}_\omega)_{\mathrm{CM}},\\ \ddot{\vec{r}} &= \frac{1}{\mu}\frac{q_1 q_2}{|r|^2}\vec{n} + \left(\frac{q_1}{m_1} - \frac{q_2}{m_2}\right)\left(\vec{E}_\omega\right)_{\mathrm{CM}}.\end{aligned}$$

Angular momentum variation:

$$\frac{d\vec{L}}{dt} = \mu\frac{2q}{M}\vec{r} \times \left(\vec{E}_\omega\right)_{\mathrm{CM}}.$$

Electromagnetic scattering framework 2

Perturbative expansion:

$$L(t) = L_\circ + E_\omega L_1(t) + \mathcal{O}\left(E_\omega^2\right) \ .$$

$$r(t) = r_\circ + E_\omega g(t) \, .$$

$$\dot{\phi} = \frac{2L(t)}{r^2 M} = \omega_0 + E_\omega Z_p .$$

Equation of motion:

$$\ddot{\vec{R}}_{\mathrm{CM}} = \frac{q_1 + q_2}{m} (\vec{E}_\omega)_{\mathrm{CM}},$$

$$\ddot{\vec{r}} = \frac{1}{\mu} \frac{q_1 q_2}{|r|^2} \vec{n} + \left(\frac{q_1}{m_1} - \frac{q_2}{m_2}\right) \left(\vec{E}_\omega\right)_{\mathrm{CM}} .$$

Angular momentum variation:

$$\frac{d\vec{L}}{dt} = \mu \frac{2q}{M} \vec{r} \times \left(\vec{E}_\omega\right)_{\mathrm{CM}} \ .$$

Electromagnetic scattering radial solution

$$\ddot{g}(t) + \left(\frac{12L_\circ^2}{M^2 r_\circ^4} - \frac{4q^2}{M r_\circ^3}\right) g(t) + \frac{8 L_\circ q\, r_\circ \omega_0 c_\gamma}{M^2 r_\circ^3(\omega_0^2 - \Omega^2)} =$$

$$\frac{4 L_\circ q\, r_\circ}{M^2 r_\circ^3}\left(\frac{c_{(\omega_0-\Omega)t-\gamma}}{\omega_0 - \Omega} + \frac{c_{(\omega_0+\Omega)t-\gamma}}{\omega_0 + \Omega}\right)$$

$$\frac{q}{M}[c_{(\omega_0+\Omega)t-\gamma} + c_{(\omega_0-\Omega)t-\gamma}]\,.$$

$$\begin{aligned}
g(t) \quad &= \quad k_2 s_{\omega_0 t} + k_1 c_{\omega_0 t} \\
&+ \quad q\left(\frac{c_{\omega_0 t}(4\omega_0 s_\gamma s_{\Omega t} - 2\Omega c_\gamma c_{\Omega t}) - s_{\omega_0 t}(4\omega_0 c_\gamma s_{\Omega t} - 2\Omega s_\gamma c_{\Omega t})}{M\Omega^3 - 4M\Omega\omega_0^2}\right) \\
&+ \quad q^3\left(\frac{4c_\gamma}{M r_\circ \Omega^2 \omega_0 - M r_\circ \omega_0^3} - \frac{4c_{\omega_0 t}\left(s_\gamma\left(\Omega^2 + 2\omega_0^2\right)s_{\Omega t} - 3\Omega\omega_0 c_\gamma c_{\Omega t}\right)}{M r_\circ \Omega\left(\Omega^4 - 5\Omega^2\omega_0^2 + 4\omega_0^4\right)}\right) \\
&+ \quad q^3\left(\frac{4s_{\omega_0 t}\left(c_\gamma\left(\Omega^2 + 2\omega_0^2\right)s_{\Omega t} + 3\Omega\omega_0 s_\gamma c_{\Omega t}\right)}{M r_\circ \Omega\left(\Omega^4 - 5\Omega^2\omega_0^2 + 4\omega_0^4\right)}\right)\,.
\end{aligned}$$

Electric field

$$
\begin{aligned}
\vec{E}_{2p}(t) &= -\frac{2q^3}{c^2 R_0 M\, r_\circ^2}\left[\left(\vec{n}(t)\times\hat{R}_0\right)\times\hat{R}_0\right] \\
&+ \frac{2q^2 E_\omega c_{\Omega t}}{c^2 R_0 M}\left[\left((\hat{E}_\omega)_{CM}\times\hat{R}_0\right)\times\hat{R}_0\right] \\
&+ \frac{4q^6 E_\omega}{c^2 R_0 M^2\, r_\circ^3}\left[\frac{4}{r_\circ\Omega^2\omega_0 - r_\circ\omega_0^3} + \frac{4\left(\Omega^2+2\omega_0^2\right) s_{\Omega t} s_{\omega_0 t}}{r_\circ\Omega\left(\Omega^4 - 5\Omega^2\omega_0^2 + 4\omega_0^4\right)}\right] \\
&+ \frac{4q^6 E_\omega}{c^2 R_0 M^2\, r_\circ^3}\left[\frac{12\omega_0 c_{\Omega t} c_{\omega_0 t}}{r_\circ\left(\Omega^4 - 5\Omega^2\omega_0^2 + 4\omega_0^4\right)}\right]\left[\left(\vec{n}(t)\times\hat{R}_0\right)\times\hat{R}_0\right] \\
&- \frac{4q^4 E_\omega}{c^2 R_0 M^2\, r_\circ^3}\left[\frac{4\omega_0 s_{\Omega t} s_{\omega_0 t}}{\Omega^3 - 4\Omega\omega_0^2} + \frac{2\Omega c_{\Omega t} c_{\omega_0 t}}{\Omega^3 - 4\Omega\omega_0^2}\right]\left[\left(\vec{n}(t)\times\hat{R}_0\right)\times\hat{R}_0\right].
\end{aligned}
$$

Cross section

Definition:

$$d\sigma = \frac{dI_{\text{scat}}}{S_\omega}.$$

$$\frac{d\sigma}{do} = \frac{\langle S_{\text{scat}} \rangle R_0^2}{\langle S_\omega \rangle},$$

Poynting vectors:

$$\vec{S}_\omega = \left(\frac{c}{4\pi} E_\omega^2 c_{\Omega t}^2 \right) \mathbf{N}.$$

$$\langle S_\omega \rangle = \frac{\Omega}{2\pi} \int_{T_\omega} |\vec{S}_\omega| = \frac{cE_\omega^2}{8\pi}.$$

Stars as resonant absorbers of gravitational waves

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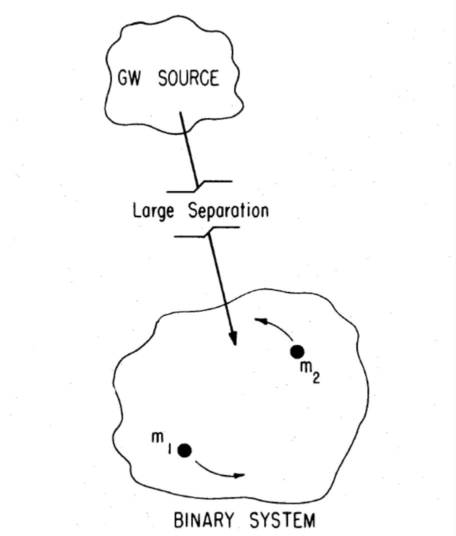


Figure 4: Sketch of the GW scattering. [Turner, 1979ApJ233]

Einstein equations

$$G^{\mu\nu} = \frac{8\pi G}{c^4} T^{\mu\nu}, \quad T^{\mu\nu} = \sum_{A=1,2} \frac{m_A}{\sqrt{-g}} \frac{v_A^\mu v_A^\nu}{\sqrt{-g_{\rho\sigma} \frac{v_A^\rho v_A^\sigma}{c^2}}} \delta^{(3)}(\mathbf{x} - \mathbf{y}_A)\,,$$

Defining the gothic metric $\mathfrak{g}^{\mu\nu} = \sqrt{-g} g^{\mu\nu}$ and the tensor $H^{\mu\alpha\nu\beta} = \mathfrak{g}^{\alpha\beta}\mathfrak{g}^{\mu\nu} - \mathfrak{g}^{\alpha\nu}\mathfrak{g}^{\beta\mu}$:

$$\partial_{\alpha\beta} H^{\mu\alpha\nu\beta} = (-g)\left(2G^{\mu\nu} + \frac{16\pi G}{c^4} t_{\mathrm{LL}}^{\mu\nu}\right)\,,$$

We can rewrite the field equations as

$$\Box h^{\mu\nu} = \frac{16\pi G}{c^4} T^{\mu\nu} + \Lambda^{\mu\nu}, \quad \Lambda^{\mu\nu} = \frac{16\pi G}{c^4} t_{\mathrm{LL}}^{\mu\nu} + \partial_\rho h^{\mu\sigma} \partial_\sigma h^{\nu\rho} - h_{\rho\sigma} \partial_{\rho\sigma} h^{\mu\nu}\,,$$

Λ is at least quadratic in the gravitational field.

Post Newtonian potentials

Parametrization of the metric by the usual PN potentials, using the variable $h^{00ii} \equiv h^{00} + h^{ii}$:

$$h^{00ii} = -\frac{4V}{c^2} + \mathcal{O}(c^{-4}), \quad h^{0i} = -\frac{4V^i}{c^3} + \mathcal{O}(c^{-5}), \quad h^{ij} = \mathcal{O}(c^{-4}).$$

Decomposition of V in a Newtonian part and a linear in H contribution, $V = V_N + V_h$.

$$V_N = \frac{Gm_1}{r_1} + \frac{Gm_2}{r_2}\,.$$

$$V_h = \Delta^{-1}\left[-H^{ab}\partial_{ab}\left(\frac{Gm_1}{r_1} + \frac{Gm_2}{r_2}\right)\right]\,.$$

Geodesic equation

$$\frac{\mathrm{d}\left(P^i\right)_1}{\mathrm{d}t} = \left(F^i\right)_1 \ ,$$

with

$$P^i = \frac{g^i_\mu v^\mu}{\sqrt{-g_{\rho\sigma} v^\rho v^\sigma}}, \quad F^i = \frac{1}{2} \frac{\partial^i g_{\mu\nu} v^\mu v^\nu}{\sqrt{-g_{\rho\sigma} v^\rho v^\sigma}}.$$

Angle-action formalism 1

Lagrangian in the center-of-mass coordinates, using spherical coordinates (r, θ, φ):

$$\tilde{L} \equiv \frac{L}{m\nu} = \frac{Gm}{r} + \frac{1}{2}\dot{r}^2 + \frac{1}{2}r^2\left(\dot{\theta}^2 + \sin^2\theta\,\dot{\varphi}^2\right)\,,$$

we determine the conjugate momenta $p_x = \partial\tilde{L}/\partial\dot{x}$ and the reduced Hamiltonian:

$$\begin{aligned}\tilde{\mathcal{H}}_0 &\equiv p_r\dot{r} + p_\theta\dot{\theta} + p_\varphi\dot{\varphi} - \tilde{L} \\ &= -\frac{Gm}{r} + \frac{1}{2}p_r^2 + \frac{1}{2r^2}p_\theta^2 + \frac{1}{2r^2\sin^2\theta}p_\varphi^2\,.\end{aligned}$$

The angular momentum $\mathbf{L} = \mathbf{r} \wedge \mathbf{v}$ is then given by

$$L_r = 0\,, \qquad L_\theta = -\frac{p_\varphi}{\sin\theta}\,, \qquad L_\varphi = p_\theta\,.$$

Angle-action formalism 2

From $(r, \theta, \varphi,\ p_r, p_\theta, p_\varphi)$ to angle-action variables using the **modified Delaunay variables**.

$$J_3 = \frac{Gm}{\sqrt{-2E}}\,, \quad J_2 = \frac{Gm}{\sqrt{-2E}} - L\,, \quad J_1 = L - L_z\,.$$

The Hamiltonian is $\tilde{\mathcal{H}}_0 = -\frac{G^2 m^2}{2J_3^2}$ and the frequencies are:

$$\Omega_3 = \frac{G^2 m^2}{J_3^3}\,, \qquad \Omega_2 = 0\,, \qquad \Omega_1 = 0\,.$$

The angles θ_i, conjugate variables of the action J_i, are then linear in time.

Delaunay variables and orbital elements

$$\begin{aligned}
J_3 &= \sqrt{Gma}, \quad J_2 = \sqrt{Gma}\left(1 - \sqrt{1-e^2}\right) , \\
J_1 &= \sqrt{Gma(1-e^2)}\,(1-\cos\iota) , \\
\theta_3 &= l + \omega + \psi , \quad \theta_2 = -(\omega+\psi) , \quad \theta_1 = -\psi .
\end{aligned}$$

Perturbative angle-action 1

$$\tilde{\mathcal{H}} = \tilde{\mathcal{H}}_0 \left(\mathbf{J}\right) + \tilde{\mathcal{H}}_1 \left(\boldsymbol{\theta}, \mathbf{J}, t\right) \ ,$$

The perturbation is small: unperturbed (Newtonian) results to evaluate $\tilde{\mathcal{H}}_1$ and its derivatives. Secular evolution obtained averaging over one orbit.

Let's define a new set of canonical angle-action coordinates $(\boldsymbol{\theta}_{\rm new},\ \boldsymbol{J}_{\rm new})$ such that the Hamiltonian will only depend on the action variables.

Perturbative angle-action 2

This transformation is valid only when $\mathbf{k}\cdot\boldsymbol{\Omega}^0(\mathbf{J})\neq 0$. The case

$$\mathbf{k}\cdot\boldsymbol{\Omega}^0(\mathbf{J}) = 0\,,$$

is called the **problem of small divisors** and it describes the appearance of a reasonance at the corresponding frequency.

$$\Omega = \pm\frac{G^2 m^2}{J_3^3}\,n\,,$$

$$\begin{aligned}
\tilde{L} &= \frac{Gm}{r}\left[1+\frac{1}{2}\,H_{ij}n^i n^j\right]+\frac{1}{2}\,\dot{r}^2\left[1-H_{ij}n^i n^j\right]\\
&+ \frac{1}{2}\,r^2\,\dot{\theta}^2\left[1-H_{ij}\theta^i\theta^j\right]+\frac{1}{2}\,r^2\,\sin^2\theta\,\dot{\varphi}^2\left[1-H_{ij}\varphi^i\varphi^j\right]\\
&- r\,\dot{r}\,\dot{\theta}\,H_{ij}n^i\theta^j - r\,\sin\theta\,\dot{r}\,\dot{\varphi}\,H_{ij}n^i\varphi^j - r^2\,\sin\theta\,\dot{\theta}\,\dot{\varphi}\,H_{ij}\theta^i\varphi^j\,,
\end{aligned}$$

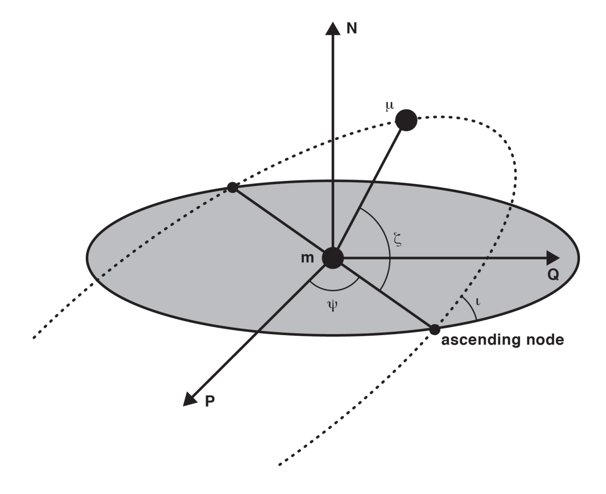
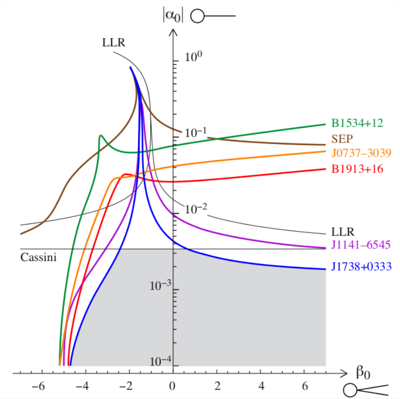


Figure 5: Three dimensional configuration of the binary system with explicit reference to the Keplerian parameter. [LA, Bernard, Blas, Cardoso (2018)]

Constraints by binary pulsars

$$\frac{\dot{P}}{P} = -\frac{8}{5}\frac{\mu m^2}{r^4}\kappa_1 - \frac{\mu m}{r^3}\kappa_D(\alpha_1 - \alpha_2)^2, \quad \kappa_D = 2\mathcal{G}\xi\frac{\omega^2 - m_\varphi^2}{\omega^2}\Theta(\omega - m_\varphi)$$



If the scalar field is massive, β doesn't obey to this limit!

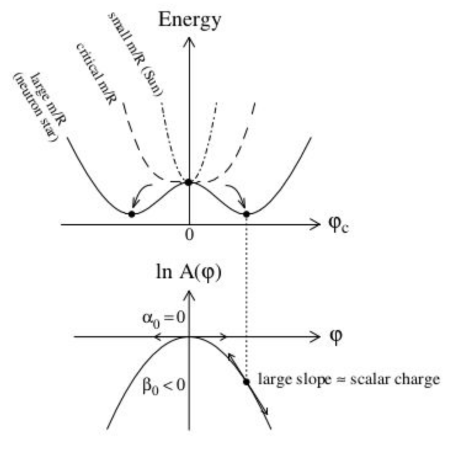


Figure 6: Spontaneous symmetry breaking in spontaneous scalarization [Esposito-Farèse, arxiv:0409081]

Vector-Tensor Einstein-like equation

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \omega \left(A_\mu A_\nu R + \Phi R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \Phi R - \Phi_{;\mu\nu} + g_{\mu\nu} \Box^2 \Phi \right)$$

TOV in vector-tensor

A system of differential equations as:

$$\begin{aligned}
m' &= m'[m(r), \epsilon(r), \nu(r), p(r), X_0(r), X_0'(r)] \\
\nu' &= \nu'[m(r), \epsilon(r), \nu(r), p(r), X_0(r), X_0'(r)] \\
p' &= p'[m(r), \epsilon(r), \nu(r), p(r), X_0(r), X_0'(r)] \\
X_0' &= X_0'[m(r), \epsilon(r), \nu(r), p(r), X_0(r), X_0'(r)]
\end{aligned}$$

Next step is search for the coefficient describing the expansion at the center of the unknown functions:

$$m \to m_3 r^3,\ p \to p_0 + p_1 r + \frac{p_2}{2} r^2$$

$$\epsilon \to \epsilon_0 + \epsilon_1 r + \frac{\epsilon_2}{2} r^2$$

$$X_0 \to X_{0,\,c} + X_{0,\,1} r + \frac{X_{0,\,2}}{2} r^2,\ \ X_0' \to X_{0,\,1} X_{0,\,2} r$$

Lovelock Theorem

"In four spacetime dimensions the only divergence-free symmetric rank-2 tensor constructed solely from the metric $g_{\mu\nu}$ and its derivatives up to second differential order, and preserving diffeomorphism invariance, is the Einstein tensor plus a cosmological term."

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu}.$$

Action in the Jordan (or Physical) Frame

$$S = \int d^4x \frac{\sqrt{-\tilde{g}}}{16\pi}[\phi\tilde{R} - \frac{\omega(\phi)}{\phi}\tilde{g}^{\mu\nu}\tilde{\nabla}_\mu\phi\tilde{\nabla}_\nu\phi - V(\phi)] + S_m[\psi_m, \tilde{g}_{\mu\nu}].$$

EoM in the Jordan-Frame (JF)

$$\begin{aligned}
\tilde{R}_{\mu\nu} - \frac{1}{2}\tilde{g}_{\mu\nu} =& \frac{8\pi}{\phi}\tilde{T}^m_{\mu\nu} + \frac{\omega(\phi)}{\phi^2}(\tilde{\nabla}_\mu\phi\tilde{\nabla}_\nu\phi - \frac{1}{2}\tilde{g}_{\mu\nu}\tilde{\nabla}^\sigma\phi\tilde{\nabla}_\sigma\phi) \\
&+ \frac{1}{\phi}(\nabla_\mu\nabla_\nu\phi - \tilde{g}_{\mu\nu}\tilde{\Box}\phi) - \frac{V(\phi)}{2\phi}\tilde{g}_{\mu\nu}, \\
(2\omega + 3)\tilde{\Box}\phi =& 8\pi\tilde{T}^m - \frac{d\omega}{d\phi}\tilde{\nabla}^\sigma\phi\tilde{\nabla}_\sigma\phi + \phi\frac{dV}{d\phi} - 2V,
\end{aligned}$$

where $\tilde{T}^{\mu\nu} = -2(-\tilde{g})^{-1/2}\delta S_m[\psi, \tilde{g}_{\mu\nu}]/\delta\tilde{g}_{\mu\nu}$.

Conformal transformation rules 1

Let's introduce the CT Ω:

$$\tilde{g}_{\mu\nu} = \Omega^{-2} g_{\mu\nu}, \quad \tilde{g}^{\mu\nu} = \Omega^{2} g^{\mu\nu}, \quad \sqrt{-\tilde{g}} = \Omega^{-4}\sqrt{-g}.$$

With this CT it is possible to show that the Ricci tensor in the JF can be expressed as a function of the new conformal metric $g_{\mu\nu}$:

$$\tilde{R} = \Omega^2 [R + 6\Box f - 6 g^{\mu\nu} f_\mu f_\nu],$$

where

$$f_\mu = \partial_\mu (\ln \Omega) = \frac{\partial_\mu \Omega}{\Omega}.$$

With scalar field (and potential) redefinition:

$$d\varphi = \sqrt{\frac{2\omega(\phi) + 3}{16\pi}} \frac{d\phi}{\phi}, \quad U(\varphi) = \frac{V(\phi)}{\phi^2}.$$

Conformal transformation rules 2

...and after some manipulations to get the usual kinetic term and defining the coupling function $A(\varphi)$, we can find:

$$S_{EF} = \int d^4x \frac{\sqrt{-g}}{16\pi}[R - \frac{1}{2}g^{\mu\nu}\nabla_\mu\varphi\nabla_\nu\varphi - U(\varphi)] + S_m[\psi_m, A^2(\varphi)g_{\mu\nu}].$$

EoM Einstein-Frame (EF)

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi T^m_{\mu\nu} + (\nabla_\mu\varphi\nabla_\nu\varphi - \frac{1}{2}g_{\mu\nu}\nabla^\sigma\varphi\nabla_\sigma\varphi) - U(\varphi)g_{\mu\nu},$$

$$\Box\varphi = -4\pi\alpha(\varphi)T + \frac{dU(\varphi)}{d\varphi},$$

where $T^{\mu\nu} = -2(-g)^{-1/2}\delta S_m[\psi, A^2(\varphi)g_{\mu\nu}]/\delta g_{\mu\nu}$.

Scalar-Tensor Frames Highlight

The **Jordan Frame** is called the physical frame, because particles follow the same geodesic as the ones defined by GR: **universality of free falling.**

In the **Einstein Frame**, since the matter is directly affected by the scalar field value, the **Strong Equivalence Principle is not satisfied.**

Ramazanoglu's vector-tensor

The action of the theory is:

$$\int d^4x\sqrt{-g}[R - F_{\mu\nu}F^{\mu\nu} - m^2 X_\mu X^\mu] + S_m[\psi_m, \Omega^{-2} g_{\mu\nu}]$$

where X_μ is the Vector field that couples with the metric tensor directly in the matter action.

The conformal transformation used is:

$$\Omega^{-2} = e^{-\beta\eta}, \text{ with } \eta = X_\mu X^\mu$$